Resit Exam — Analysis (WBMA012-05)

Thursday 10 April 2025, 8.20h–10.30h

University of Groningen

Instructions

1. The use of calculators, books, or notes is not allowed.
2. Provide clear arguments for all your answers: only answering "yes", "no", or "42" is not sufficient. You may use all theorems and statements in the book, but you should clearly indicate which of them you are using.
3. The total score for all questions equals 90. If p is the number of marks then the exam grade is $G = 1 + p/10$.

Problem 1 (7 + 4 + 5 = 16 points)

(a) Prove that $\sqrt{5}$ is irrational.

(b) Show that for all $n \in \mathbb{N}$ there exisists $a_n \in \mathbb{Q}$ such that

$$\sqrt{5} < a_n < \sqrt{5} + \frac{1}{n}.$$

(c) Explain why $\mathbb{Q}$ does *not* satisfy the Axiom of Completeness.

Problem 2 (6 + 5 + 10 + 5 = 26 points)

Let (a_n) be a bounded sequence and define the sequences (y_n) and (z_n)

$$y_n = \sup\{a_k \mid k \geq n\} \quad \text{and} \quad z_n = \inf\{a_k \mid k \geq n\}.$$

Furthermore, define

$$\limsup a_n = \lim y_n \quad \text{and} \quad \liminf a_n = \lim z_n.$$

(a) Prove that the sequences (y_n) and (z_n) are convergent.

(b) Prove that $\liminf a_n \leq \limsup a_n$.

(c) Show that $\lim a_n$ exists *if and only if* $\liminf a_n = \limsup a_n$, and in this case

$$\liminf a_n = \limsup a_n = \lim a_n.$$

(d) Give an example for which the inequality $\liminf a_n < \limsup a_n$ is strict.

Problem 3 (16 points)

Let $A, B \subseteq \mathbb{R}$ two arbitrary sets. We define their sum as

$$A + B = \{a + b \mid a \in A, b \in B\}.$$

Prove that if A and B are both compact, then $A + B$ is compact.
Hint: use the definition of compactness

Please turn over for problems 4 and 5!

Problem 4 (4 + 12 = 16 points)

Let f be a continuous function on an interval I and O be the set of points where f fails to be injective, that is,

$$O = \{x \in I \mid \exists y \in I,\ y \neq x, \text{ such that } f(x) = f(y)\} \subset I.$$

Assume that O is not empty, and prove the following statements:

(a) If f is constant in a sub-interval $[a, b] \subset I$, then O is uncountable.

(b) If f is not constant in I, then O is uncountable.
Hint: reason with the Intermediate Value Theorem.

Problem 5 (4 + 12 = 16 points)

(a) Argue that the function $f(x) = 1/x$ is integrable on $[1, 2]$

(b) Use the partition $P = \left\{\frac{k+n}{n} \mid k = 0, \ldots, n\right\}$ to prove the following inequality:

$$\ln 2 \leq \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}$$

for all $n \in \mathbb{N}$

End of test (90 points)

Note that all the problems could be solved in multiple ways, and not all of those solutions are included here.

Solution of Problem 1 (7 + 4 + 5 = 16 points)

(a) We use proof by contradiction. Assume $\sqrt{5}$ is rational. Then there exist integers integers p and q with $q \neq 0$ such that

$$\sqrt{5} = \frac{p}{q} \quad \Leftrightarrow \quad p^2 = 5q^2.$$

We can assume that the fraction $\frac{p}{q}$ is in lowest terms, meaning $\gcd(p, q) = 1$.

Since the right hand side is a muyltiple of 5, p^2 is divisible by 5. Since 5 is a prime number, if it divides p^2, it must also divide p as all its factors have to appear repeated in the square. So, $p = 5k$ for some integer k.

Substituting $p = 5k$ into the equation we get

$$(5k)^2 = 5q^2 \quad \Leftrightarrow 5k^2 = q^2.$$

As above, since the left hand side is a multiple of 5, this implies that q^2 is divisible by 5, and, thus, q must also be divisible by 5.

So, both p and q have factor 5 in common. This contradicts our initial assumption that $\frac{p}{q}$ is in lowest terms. Therefore, $\sqrt{5}$ must be irrational.

(b) Since rational numbers are dense in the real numbers, for all $a, b \in \mathbb{R}$ there exists a number $r \in \mathbb{Q}$ such that $a < r < b$.

Applying this with $b = \sqrt{5} + 1/n$ and $a = \sqrt{5}$ gives the final statement with $a_n = r$.

(c) Reminder: The Axiom of Completeness states that every non-empty subset of $\mathbb{R}$ that is bounded above has a least upper bound (supremum) in $\mathbb{R}$. We want to show that $\mathbb{Q}$ does not satisfy this axiom, that is, that we cannot simply replace $\mathbb{R}$ with $\mathbb{Q}$. To this end, we need to find a non-empty subset of $\mathbb{Q}$ that is bounded above, but does not have a supremum in $\mathbb{Q}$.

Method 1

The same reasoning as in point (b) shows that for all $n \in \mathbb{N}$ there exists b_n such that $\sqrt{5} - 1/n < b_n < \sqrt{5}$.

Let $B = \{b_n \mid n \in \mathbb{N}\}$. Then B is bounded above, since $b_n < \sqrt{5}$ for all n.

Furthermore, B does not have a least upper bound in $\mathbb{Q}$. From (b) it follows that $\sup B = \sqrt{5}$ and $\sqrt{5} \notin \mathbb{Q}$. So not all sets in $\mathbb{Q}$ that are bounded above have a least upper bound in $\mathbb{Q}$, and thus the Axiom of Completeness does not hold for $\mathbb{Q}$.

Method 2

In the Analysis exam, we saw that this is equivalent to state that every non-empty subset of the reals bounded below has a greatest lower bound in $\mathbb{R}$. With this in mind, one can consider the set $A = \{a_n \mid n \in \mathbb{N}\}$ with a_n from point (b).

Then $A \subset \mathbb{Q}$ is bounded below, since $\sqrt{5} < a_n$ for all n.

Furthermore, A does not have a greatest lower bound in $\mathbb{Q}$. From (b) it follows that $\inf A = \sqrt{5}$ and $\sqrt{5} \notin \mathbb{Q}$. So not all sets in $\mathbb{Q}$ that are bounded below have a greatest lower bound in $\mathbb{Q}$, and thus the Axiom of Completeness does not hold for $\mathbb{Q}$.

Solution of Problem 2 (6 + 5 + 10 + 5 = 26 points)

(a) Let $A_n = \{a_k \mid k \geq n\}$. Then $y_n = \sup A_n$ and $z_n = \inf A_n$. Since (a_n) is bounded, there exist real numbers m and M such that $m \leq a_k \leq M$ for all $k \in \mathbb{N}$.

Consider the sequence (y_n). For any n, A_n is non-empty and bounded above by M. By the Axiom of Completeness, $y_n = \sup A_n$ exists for all n.

Also, $A_{n+1} = \{a_k \mid k \geq n+1\} \subseteq \{a_k \mid k \geq n\} = A_n$. Therefore, $\sup A_{n+1} \leq \sup A_n$, that is, $y_{n+1} \leq y_n$. So, the sequence (y_n) is monotonically decreasing.

Since $m \leq a_k$ for all k, m is a lower bound for every set A_n. Thus, $y_n = \sup A_n \geq m$ for all n. So, (y_n) is monotonically decreasing and bounded below by m.

By the Monotone Convergence Theorem, a monotonically decreasing sequence that is bounded below converges. Therefore, (y_n) converges.

The convergence of z_n is similar. One can argue that $(-z_n)$ converges exactly for the same reasons explained above, one can explain how the reasoning has to differ from the above, or one can also do it all explicitly as shown below.

Consider the sequence (z_n). For any n, A_n is non-empty and bounded below by m. By the Axiom of Completeness (applied to $-A_n$), $z_n = \inf A_n$ exists for all n.

Since $A_{n+1} \subseteq A_n$, $\inf A_{n+1} \geq \inf A_n$ So, $z_{n+1} \geq z_n$. Thus, the sequence (z_n) is monotonically increasing.

Since $a_k \leq M$ for all k, M is an upper bound for every set A_n. Thus, $z_n = \inf A_n \leq M$ for all n. So, (z_n) is monotonically increasing and bounded above by M.

By the Monotone Convergence Theorem, a monotonically increasing sequence that is bounded above converges. Therefore, (z_n) converges.

(b) Using the notation introduced above,by the definition of infimum and supremum for a non-empty set A_n, we know that $\inf A_n \leq \sup A_n$. So, $z_n \leq y_n$ for all $n \in \mathbb{N}$.

In part (a), we proved that both (y_n) and (z_n) converge. Let $L_y = \lim y_n = \limsup a_n$ and $L_z = \lim z_n = \liminf a_n$.

Since $z_n \leq y_n$ for all n, by the Order Limit Theorem (which states that if $x_n \leq y_n$ for all n and both limits exist, then $\lim x_n \leq \lim y_n$), we can take the limit as $n \to \infty$: $L_z = \lim z_n \leq \lim y_n = L_y$. Therefore, $\liminf a_n \leq \limsup a_n$.

(c) ($\Rightarrow$) Assume $\lim a_n = L$ exists. This means that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $k \geq N$, $|a_k - L| < \epsilon$, which implies $L - \epsilon < a_k < L + \epsilon$.

For $n \geq N$, the set $A_n = \{a_k \mid k \geq n\}$ satisfies $L - \epsilon < a_k < L + \epsilon$ for all its elements. This means $L + \epsilon$ is an upper bound for A_n, so $y_n = \sup A_n \leq L + \epsilon$.

Also, $L - \epsilon$ is a lower bound for A_n, so $z_n = \inf A_n \geq L - \epsilon$. Thus, for all $n \geq N$, we have $L - \epsilon \leq z_n \leq y_n \leq L + \epsilon$.

Since $\lim z_n = \liminf a_n$ and $\lim y_n = \limsup a_n$, taking the limit as $n \to \infty$ gives:

$$L - \epsilon \leq \liminf a_n \leq \limsup a_n \leq L + \epsilon.$$

This holds for any $\epsilon > 0$ and thus $L = \liminf a_n \leq \limsup a_n = L$, concluding the proof.

($\Leftarrow$) Assume $\liminf a_n = \limsup a_n = L$. Let $y = \lim y_n = L$ and $z = \lim z_n = L$. We want to show that $\lim a_n = L$. Let $\epsilon > 0$.

Since $\lim y_n = L$, there exists $N_y \in \mathbb{N}$ such that for all $n \geq N_y$, $|y_n - L| < \epsilon$. Since (y_n) is decreasing to L, this implies $L \leq y_n < L + \epsilon$.

Since $\lim z_n = L$, there exists $N_z \in \mathbb{N}$ such that for all $n \geq N_z$, $|z_n - L| < \epsilon$. Since (z_n) is increasing to L, this implies $L - \epsilon < z_n \leq L$.

Let $N = \max(N_y, N_z)$. Then for all $n \geq N$, we have

$$L - \epsilon < z_n \leq L \leq y_n < L + \epsilon.$$

By definition, $z_n = \inf\{a_k \mid k \geq n\}$ and $y_n = \sup\{a_k \mid k \geq n\}$. This means that for any $k \geq n$, we have $z_n \leq a_k \leq y_n$.

Now, let $k \geq N$. We can apply the above with $n = k$. We have $z_k \leq a_k \leq y_k$. Since $k \geq N$, we know $L - \epsilon < z_k$ and $y_k < L + \epsilon$. Combining these inequalities, we get

$$L - \epsilon < z_k \leq a_k \leq y_k < L + \epsilon.$$

This implies $L - \epsilon < a_k < L + \epsilon$, or $|a_k - L| < \epsilon$.

Since this holds for all $k \geq N$, by definition of limit, $\lim a_k = L$ concluding the proof.

(d) Consider the sequence $a_n = (-1)^n$. The sequence terms are $-1, 1, -1, 1, \ldots$.

The sequence is bounded, as $|a_n| = 1 \leq 1$ for all n. Moreover, for all n, we have $y_n = \sup A_n = \sup\{-1, 1\} = 1$ and $z_n = \inf A_n = \inf\{-1, 1\} = -1$.

Thus, $\liminf a_n = -1$ and $\limsup a_n = 1$ showing that $\liminf a_n < \limsup a_n$.

Solution of Problem 3 (16 points)

We will use the sequential definition of compactness: A set $K \subseteq \mathbb{R}$ is compact if and only if every sequence in K has a subsequence that converges to a point in K.

Let (c_n) be an arbitrary sequence in $A+B$. We want to show that (c_n) has a subsequence that converges to a point in $A+B$.

By the definition of $A+B$, there are a sequence (a_n) in A and a sequence (b_n) in B, such that $c_n = a_b + b_n$ for all $n \in \mathbb{N}$.

Since A is compact, the sequence (a_n) contains a subsequence (a_{n_k}) that converges to some $a \in A$. That is, $\lim_{k\to\infty} a_{n_k} = a \in A$.

Consider the corresponding subsequence (b_{n_k}) of (b_n). Since B is compact, the sequence (b_{n_k}) must contain a subsequence $(b_{n_{k_j}})$ that converges to some $b \in B$. That is, $\lim_{j\to\infty} b_{n_{k_j}} = b \in B$.

At this point we can further restrict (a_n) and consider the subsequence $(a_{n_{k_j}})$ of (a_{n_k}).

Since (a_{n_k}) converges to a, any subsequence of it must also converge to the same limit a. So, $\lim_{j\to\infty} a_{n_{k_j}} = a$.

Now consider the subsequence $(c_{n_{k_j}})$ of the original sequence (c_n). We have $c_{n_{k_j}} = a_{n_{k_j}} + b_{n_{k_j}}$.

By the Algebraic Limit Theorem, the limit of the sum is the sum of the limits (provided the individual limits exist) it follows that $(c_{n_{k_j}})$ is convergent with

$$c = \lim_{j\to\infty} c_{n_{k_j}} = \lim_{j\to\infty} (a_{n_{k_j}} + b_{n_{k_j}}) = \lim_{j\to\infty} a_{n_{k_j}} + \lim_{j\to\infty} b_{n_{k_j}} = a + b \in A + B.$$

This concludes the proof since we have shown that every sequence (c_n) in $A+B$ has a convergent subsequence $(c_{n_{k_j}})$ whose limit is again in $A+B$.

Alternative proof using Heine-Borel Theorem

In $\mathbb{R}$, a set is compact if and only if it is closed and bounded. A and B are compact, so they are closed and bounded.

Boundedness of $A+B$: Since A is bounded, there exists $M_A > 0$ such that $|a| \leq M_A$ for all $a \in A$. Since B is bounded, there exists $M_B > 0$ such that $|b| \leq M_B$ for all $b \in B$. Let $c \in A+B$. Then $c = a+b$ for some $a \in A$ and $b \in B$. By the triangle inequality, $|c| = |a+b| \leq |a| + |b| \leq M_A + M_B$. Let $M = M_A + M_B$. Then $|c| \leq M$ for all $c \in A+B$. Thus, $A+B$ is bounded.

Closedness of $A+B$: Let (c_n) be a sequence in $A+B$ such that $c_n \to c$. We need to show that $c \in A+B$. For each n, $c_n = a_n + b_n$ for some $a_n \in A$ and $b_n \in B$. The sequence (a_n) is in A. Since A is compact, it is bounded. By Bolzano-Weierstrass theorem, there exists a convergent subsequence (a_{n_k}) of (a_n). Let $a_{n_k} \to a$. Since A is closed, the limit point a must be in A. Now consider the corresponding subsequence (c_{n_k}). Since (c_n) converges to c, its subsequence (c_{n_k}) also converges to c. So, $c_{n_k} \to c$. We have $b_{n_k} = c_{n_k} - a_{n_k}$. By the algebraic limit theorem, the sequence (b_{n_k}) converges: $\lim_{k\to\infty} b_{n_k} = \lim_{k\to\infty}(c_{n_k} - a_{n_k}) = c - a$. Let $b = c - a$. Since each b_{n_k} is in B, and B is closed, the limit point b must be in B. So, $b \in B$. We have shown that $c = a + b$ where

$a \in A$ and $b \in B$. By definition of $A + B$, this means $c \in A + B$. Therefore, $A + B$ contains all its limit points, so $A + B$ is closed.

Since $A + B$ is both closed and bounded, by the Heine-Borel Theorem, $A + B$ is compact.

Solution of Problem 4 (4 + 12 = 16 points)

(a) Assume $f(x) = c$ for all $x \in [a, b]$, where $[a, b]$ is a sub-interval of I with $a < b$. Then f is clearly not injective in $[a, b]$ since for any two values $x \neq y \in [a, b]$, we have $c = f(x) = f(y)$.

By the definition of O, this means that every $x \in [a, b]$ belongs to the set O. So, we have $[a, b] \subseteq O$.

Since O contains an uncountable set $[a, b]$, O itself must be uncountable.

(b) We are given that O is not empty. This means there exist at least two distinct points $x, y \in I$ such that $f(x) = f(y)$ and $x \neq y$.

Assume without loss of generality that $x < y$ and pick $z \in (x, y)$ such that $f(z) \neq f(x)$. If this does not exist it would mean that f is constant in (x, y) and we would be in the case already proven in (a).

By the Intermediate Value Theorem for every $L \in (f(x), f(z))$ there exists $\tilde{x} \in (x, z)$ such that $f(\tilde{x}) = L$.

Since by continuity it is also the case that $L \in (f(z), f(y))$, by the IVT there exists $\tilde{y} \in (z, y)$ such that $f(\tilde{y}) = L$.

And thus $f(\tilde{y}) = f(\tilde{x}) = L$.

Therefore f is not injective at all values $L \in (f(x), f(z))$, which is a real interval and thus uncountable.

Solution of Problem 5 (4 + 12 = 16 points)

(a) **Method 1**: the function is decreasing in $[1, 2]$. In the lectures we have shown that decreasing functions are integrable.

Method 2: the function is continuous in $[1, 2]$. In the lectures we have shown that continuouos functions on bounded intervals are integrable.

(b) Let $f(x) = 1/x$. We established in part (a) that f is integrable on $[1, 2]$. Since for $F(x) = \ln x$ we have $F'(x) = 1/x$, by the Fundamental Theorem of Calculus

$$\int_1^2 \frac{1}{x}\, dx = \ln 2 - \ln 1 = \ln 2.$$

Since f is decreasing, it follows that

$$M_k := \sup\{f(x) \mid x \in [x_{k-1}, x_k]\} = f(x_{k-1}).$$

For the partition $P = \left\{\frac{k+n}{n} \mid k = 0, \dots, n\right\}$ we then get the following upper sum:

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) \tag{1}$$

$$= \sum_{k=1}^{n} f(x_{k-1})(x_k - x_{k-1}) \tag{2}$$

$$= \sum_{k=1}^{n} \frac{n}{k-1+n}\left(\frac{k+n}{n} - \frac{k-1+n}{n}\right) \tag{3}$$

$$= \sum_{k=1}^{n} \frac{1}{n+k-1} = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}. \tag{4}$$

By the definition of the Riemann integral, for any partition P, the integral $\int_1^2 f(x)dx$ is less than or equal to the upper Riemann sum $U(f, P)$, thus

$$\ln 2 = \int_1^2 \frac{1}{x} dx \le U(f, P) = \frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n-1}.$$

Note that this inequality holds for all $n \in \mathbb{N}$.